Log-Theta Lattice: Symmetries and Indeterminacies

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Introduction

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Goal Give a "good description" of an obj. obtained by considering the arith. divisor det'd by the zero locus of the collection of theta values

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$$

— where $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} q_{\underline{v}}^{1/2l}$; $j \in \{1, 2, \ldots, l^{*} \stackrel{\mathrm{def}}{=} \frac{l-1}{2}\}$ — that makes sense from the point of view of an "alien arithmetic holomorphic structure", i.e., the ring/scheme structure of a Hodge theater related to a given Hodge theater by means of a non-ring/scheme-theoretic "link".

Alien arithmetic holomorphic structure

k: a p-adic local field $(p \neq 2)$ $\subseteq$ $\overline{k}$: an algebraic closure

$\mathcal{O}_{\overline{k}}$: the ring of integers $\supseteq$ $\mathcal{O}^{\times}_{\overline{k}}$: the group of units

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The degree of "alienness" depends on information which are shared.

Mono-anabelian transport

$$G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k) \ \curvearrowright \ \Lambda(\overline{k}) \ \stackrel{\mathrm{def}}{=} \ \varprojlim_n (\mathcal{O}^{\times}_{\overline{k}})_{\mathrm{tor}}[n] \ (= \widehat{\mathbb{Z}}(1))$$

Let $(G \curvearrowright M) \ \cong \ (G_k \curvearrowright \mathcal{O}^{\times}_{\overline{k}})$. Write $\Lambda(M) \ \stackrel{\mathrm{def}}{=} \ \varprojlim_n M_{\mathrm{tor}}[n]$.

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Theorem

$^{\exists}$*functorial algorithm*

$$G \longmapsto \mathcal{O}^{\times}_{\overline{k}}(G), \ \ \Lambda(G)$$

corresponding to $\mathcal{O}^{\times}_{\overline{k}}$, $\Lambda(\overline{k})$. *Moreover,* $^{\exists}$*functorial algorithm*

$$(G \curvearrowright M) \longmapsto \text{the } \widehat{\mathbb{Z}}^{\times}\text{-orbit of } \Lambda(M) \xrightarrow{\sim} \Lambda(G)$$

corresponding to the (nat'l) cyclotomic rigidity isom $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G_k)$.

Note: $G \curvearrowright (1 \to M_{\mathrm{tor}}[n] \to M \stackrel{\times n}{\to} M \to 1)$ induces an embedding

$$M \hookrightarrow {}_{\infty}H^1(G, \Lambda(M)) \stackrel{\mathrm{def}}{=} \varinjlim_{J \subseteq G:\ \text{open}} H^1(J, \Lambda(M)).$$

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In a similar vein, we have an embedding

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$$(G \curvearrowright M) \longmapsto \textit{the } \widehat{\mathbb{Z}}^{\times}\textit{-orbit of } M \overset{\sim}{\to} \mathcal{O}^{\times}_{\overline{k}}(G)$$

corresponding to the (nat'l) Kummer isom $\mathcal{O}^{\times}_{\overline{k}} \overset{\sim}{\to} \mathcal{O}^{\times}_{\overline{k}}(G_k)$.

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These indeterminacies "$\mathrm{Aut}(G_k)$", "$\widehat{\mathbb{Z}}^{\times}$" correspond to the indeterminacies (Ind1), (Ind2), respectively, appearing in IUT.

Mutiradiality

Let ${}^{\dagger}\mathcal{HT}$, ${}^{\dagger}\mathcal{HT}$ be two copies of a given [$\Theta^{\pm \mathrm{ell}} NF$-]Hodge theater.

In IUT, we consider the Θ-link

$$ {}^{\dagger}\mathcal{HT} \longrightarrow {}^{\ddagger}\mathcal{HT} $$

where the link is not arising from sch-/ring- theory like a "frobenius" $q \mapsto q^N$ $(q \in \mathcal{O}_k,\ N > 2)$.

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Suppose: For $\Box \in \{\dagger,\ \ddagger\}$, $^{\exists}$functorial algorithm ${}^{\Box}\Xi$

$$ {}^{\Box}\mathcal{HT} \quad \longmapsto \quad \text{data which are related to the } \Box\text{-}\Theta\text{-pilot object} $$

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Note: If the link is "isom" arising from sch-/ring- theory, then

$$(\dagger\text{-data}) \xrightarrow{\sim} (\ddagger\text{-data})$$

Since the link does not arise from sch-/ring- theory, so, a priori:

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In IUT, we often use objects isomorphic to

$$G_k; \quad (G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\times\boldsymbol{\mu}} \overset{\text{def}}{=} \mathcal{O}_{\overline{k}}^{\times}/(\mathcal{O}_{\overline{k}}^{\times})_{\mathsf{tor}})$$

as a coric object. For instance, how about the pair (isomorphic to)

$$(G_k \curvearrowright \mathcal{O}_{\overline{k}}^{\rhd} \overset{\text{def}}{=} \mathcal{O}_{\overline{k}} \setminus \{0\}) \text{ ?}$$

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$$({}^{\dagger}G_k \curvearrowright {}^{\dagger}\mathcal{O}^{\triangleright}_{\overline{k}}) \xrightarrow{\sim} ({}^{\ddagger}G_k \curvearrowright {}^{\ddagger}\mathcal{O}^{\triangleright}_{\overline{k}})$$

$$\Longrightarrow \quad {}^{\dagger}\mathcal{O}^{\triangleright}_{k} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\triangleright}_{k}$$

$$\Longrightarrow \quad {}^{\dagger}\mathbb{N} \xrightarrow{\sim} {}^{\dagger}\mathcal{O}^{\triangleright}_{k}/{}^{\dagger}\mathcal{O}^{\times}_{k} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\triangleright}_{k}/{}^{\ddagger}\mathcal{O}^{\times}_{k} \xrightarrow{\sim} {}^{\dagger}\mathbb{N}\ ;\ 1 \ \mapsto \ 1$$

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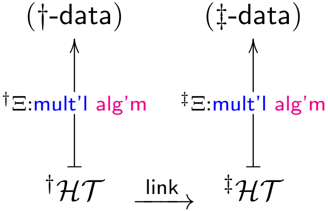
$${}^{\Box}\mathcal{HT} \quad \longmapsto \quad \text{a } \Box\text{-coric object } {}^{\Box}C \quad (\text{e.g., a top. gp } {}^{\Box}G \cong G_k)$$

Then the multiradiality of the functorial algorithms $^{\dagger}\Xi$, $^{\ddagger}\Xi$ implies:



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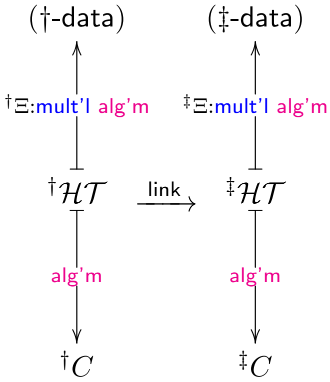

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\uparrow \; {}^{\dagger}\Xi\text{:mult'l alg'm} & & \uparrow \; {}^{\ddagger}\Xi\text{:mult'l alg'm} \\
{}^{\dagger}\mathcal{HT} & \xrightarrow{\text{link}} & {}^{\ddagger}\mathcal{HT} \\
\downarrow \; \text{alg'm} & & \downarrow \; \text{alg'm} \\
{}^{\dagger}C & \xrightarrow{\sim} & {}^{\ddagger}C
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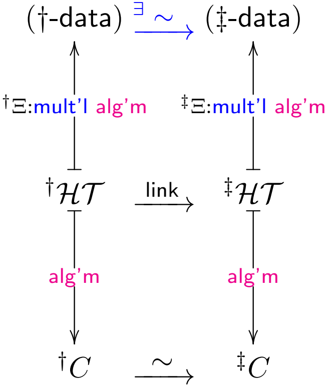

Some of notations

$(\overline{F}/F,\ E,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\mathbb{V}}^{\mathrm{bad}},\ \underline{\epsilon})$: an initial Θ-data, where

l: a prime number ≥ 5

F: a number field such that $\sqrt{-1} \in F \ \hookrightarrow \ \overline{F}$: an alg. closure

E: an elliptic curve over F that has stable reduction at all $v \in \mathbb{V}(F)^{\mathrm{non}}$

$K \overset{\mathrm{def}}{=} F(E[l]) \ \supseteq \ F \ \supseteq \ F_{\mathrm{mod}}$: the field of moduli of E

$\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$: the image of a splitting of $\mathbb{V}(K) \to \mathbb{V}(F_{\mathrm{mod}})$

$(\Longrightarrow \ \underline{\mathbb{V}} = \underline{\mathbb{V}}^{\mathrm{bad}} \cup \underline{\mathbb{V}}^{\mathrm{good}}) \ \ldots$

$X \overset{\mathrm{def}}{=} E \setminus \{o\}$: the hyperbolic curve over F assoc. to E

$\underline{X}_K \to X_K \overset{\mathrm{def}}{=} X \times_F K$: a certain finite étale covering of degree l

We consider

if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ $\Rightarrow$ $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \overset{\mathrm{def}}{=} \underline{X}_K \times_K K_{\underline{v}}$, i.e., "$\underline{\underline{X}}^{\log} \to \underline{X}^{\log}$"

if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ $\Rightarrow$ a certain finite étale covering $\underset{\rightarrow}{X}_{\underline{v}} \to \underline{X}_{\underline{v}}$

$$\Pi_{\underline{v}} \overset{\mathrm{def}}{=} \begin{cases} \pi_1^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \\ \pi_1^{\text{ét}}(\underset{\rightarrow}{X}_{\underline{v}}) & \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}, \text{ finite} \end{cases}$$

$\overline{F}_{\underline{v}}$: the algebraic closure of $K_{\underline{v}}$ det'd by $\underline{v}$ and $\overline{F}$ (up to conj.)

$G_{\underline{v}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_{\underline{v}}/K_{\underline{v}})$: the absolute Galois group of $K_{\underline{v}}$

if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ $\Rightarrow$ $\mathcal{O}_{\overline{F}_{\underline{v}}}$: the ring of integers $\supseteq \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \overset{\mathrm{def}}{=} \mathcal{O}_{\overline{F}_{\underline{v}}} \setminus \{0\}$

$\supseteq \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$: the group of units $\twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \overset{\mathrm{def}}{=} \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$

Theta monoid $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$

Let us recall the Frobenius-like/étale-like theta monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and the Kummer isomorphisms bewteen them:

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$$\Psi_{\mathcal{F}^{\Theta}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \overset{\sim}{\to} (\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \overset{\sim}{\to} \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\Pi_{\underline{v}})) \overset{\sim}{\leftarrow}) \Psi_{\Theta}({}^{\dagger}\Pi_{\underline{v}})$$

$${}_{\infty}\Psi_{\mathcal{F}^{\Theta}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \overset{\sim}{\to} ({}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \overset{\sim}{\to} {}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\Pi_{\underline{v}})) \overset{\sim}{\leftarrow}) {}_{\infty}\Psi_{\Theta}({}^{\dagger}\Pi_{\underline{v}})$$

(cf. the cyclotomic rigidity of mono-theta environments).

We consider the following radial environment:

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(cf. the cyclotomic rigidity of mono-theta environments).

We consider the following radial environment:

$\mathcal{C}$ (coric category) — Obj: $(G \curvearrowright \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}}) \cong (G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}, \{\mathcal{I}^{\kappa}_{H}\}_{H \subseteq G_{\underline{v}}})$

Hom: an isom between "$(G \curvearrowright \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}})$" that is comp. w/ "$\{\mathcal{I}^{\kappa}_{H}\}_{H \subseteq G_{\underline{v}}}$"

$\mathcal{R}$ (radial category) — Obj: data consists of the following:

(a$_{\text{ét}}$) ${}^{\dagger}\Pi_{\underline{v}} \;\cong\; \Pi_{\underline{v}}$;

(b$_{\text{ét}}$) the étale-like cyclotome ${}^{\dagger}\Pi_{\underline{v}} \curvearrowright (l \cdot \Delta_{\Theta})({}^{\dagger}\Pi_{\underline{v}})$;

(c$_{\text{ét}}$) the étale-like unit groups ${}^{\dagger}\Pi_{\underline{v}}(\twoheadrightarrow {}^{\dagger}G_{\underline{v}}) \curvearrowright (\mathcal{O}^{\times}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})})$;

(d$_{\text{ét}}$) the étale-like theta monoids $\Psi_{\Theta}({}^{\dagger}\Pi_{\underline{v}})$, ${}_{\infty}\Psi_{\Theta}({}^{\dagger}\Pi_{\underline{v}})$;

(e$_{\text{ét}}$) the canonical splitting

$$\left\{\left(\mathcal{O}^{\times}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}({}^{\dagger}\Pi_{\underline{v}})\right)/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})} \;=\; \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})} \times \left({}_{\infty}\underline{\underline{\theta}}^{\iota}({}^{\dagger}\Pi_{\underline{v}})/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})}\right)\right\}_{(\iota,D)};$$

($\mathsf{a}_{\mathsf{env}}$) the mono-theta environment $\mathbb{M}^{\Theta}_{*} \overset{\mathrm{def}}{=} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\Pi_{\underline{v}})$;

($\mathsf{b}_{\mathsf{env}}$) the exterior cyclotome ${}^{\dagger}\Pi_{\underline{v}} \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$;

($\mathsf{c}_{\mathsf{env}}$) the unit groups ${}^{\dagger}\Pi_{\underline{v}}(\twoheadrightarrow {}^{\dagger}G_{\underline{v}}) \curvearrowright (\overline{\mathcal{O}}^{\times}_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \twoheadrightarrow \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}}(\mathbb{M}^{\Theta}_{*}))$;

($\mathsf{d}_{\mathsf{env}}$) the mono-theta theoretic theta monoids $\Psi_{\Theta}(\mathbb{M}^{\Theta}_{*})$, ${}_{\infty}\Psi_{\Theta}(\mathbb{M}^{\Theta}_{*})$;

($\mathsf{e}_{\mathsf{env}}$) the canonical splitting

$$\left\{\left(\overline{\mathcal{O}}^{\times}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})\cdot{}_{\infty}\underline{\underline{\theta}}^{\iota}(\mathbb{M}^{\Theta}_{*})\right)/\overline{\mathcal{O}}^{\boldsymbol{\mu}}_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) = \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})\times\left({}_{\infty}\underline{\underline{\theta}}^{\iota}(\mathbb{M}^{\Theta}_{*})/\overline{\mathcal{O}}^{\boldsymbol{\mu}}_{\underline{v}}(\mathbb{M}^{\Theta}_{*})\right)\right\}_{(\iota,D)}$$

(f) the cyclotomic rigidity isom. ($\mathsf{b}_{\text{ét}}$) $\overset{\sim}{\to}$ ($\mathsf{b}_{\mathsf{env}}$) (cf. ($\mathsf{a}_{\mathsf{env}}$));

(g) ($\mathsf{c}_{\text{ét}}$) $\overset{\sim}{\to}$ ($\mathsf{c}_{\mathsf{env}}$), ($\mathsf{d}_{\text{ét}}$) $\overset{\sim}{\to}$ ($\mathsf{d}_{\mathsf{env}}$), ($\mathsf{e}_{\text{ét}}$) $\overset{\sim}{\to}$ ($\mathsf{e}_{\mathsf{env}}$) (cf. (f));

(h) $(G \curvearrowright \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}}) \cong (G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}, \{\mathcal{I}^{\kappa}_{H}\}_{H \subseteq G_{\underline{v}}})$;

(i) an isom. $(G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}(^{\dagger}\Pi_{\underline{v}})}) \xrightarrow{\sim} (G \curvearrowright \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}})$ (cf. ($c_{\text{ét}}$), (h));

(j) the diagram

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \overline{\mathcal{O}}^{\boldsymbol{\mu}}_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \overset{(g)}{\xrightarrow{\sim}} \mathcal{O}^{\boldsymbol{\mu}}_{\overline{k}(^{\dagger}\Pi_{\underline{v}})} \xrightarrow{\text{zero}} \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{k}(^{\dagger}\Pi_{\underline{v}})} \overset{(i)}{\xrightarrow{\sim}} \overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}}.$$

Hom: omit

(h) $(G \curvearrowright \overline{\mathcal{O}}^{\times \boldsymbol{\mu}}_{\underline{v}}) \;\cong\; (G_{\underline{v}} \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}},\ \{\mathcal{I}^{\kappa}_{H}\}_{H \subseteq G_{\underline{v}}})$;

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$\Longrightarrow$ Since the functor $\Phi : \mathcal{R} \to \mathcal{C}$ obtained by "forget. all except (h)" is full, we obtain a multiradial environment $(\mathcal{R}, \mathcal{C}, \Phi)$.

In particular, [the "functor" det'd by] the algorithm

$$^{\dagger}\Pi_{\underline{v}} \quad \mapsto \quad \Psi_{\Theta}(^{\dagger}\Pi_{\underline{v}}) \ \overset{\sim}{\to} \ \Psi_{\Theta}(\mathbb{M}^{\Theta}_{*}), \quad {}_{\infty}\Psi_{\Theta}(^{\dagger}\Pi_{\underline{v}}) \overset{\sim}{\to} {}_{\infty}\Psi_{\Theta}(\mathbb{M}^{\Theta}_{*})$$

— whose output data are appearing in the above Kummer isoms

$$\Psi_{\mathcal{F}^{\Theta}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \ \overset{\sim}{\to} \ \Psi_{\Theta}(^{\dagger}\Pi_{\underline{v}}), \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta}}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \ \overset{\sim}{\to} \ {}_{\infty}\Psi_{\Theta}(^{\dagger}\Pi_{\underline{v}})$$

— may be regarded as a multiradial algorithm.

Gaussian monoid $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot (\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,2,\dots,l^*})^{\mathbb{N}}$

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— where $\xi \in \prod_{|t| \in T^*} (\mathcal{O}^{\triangleright}_{\overline{k}({}^{\dagger}\Pi_{\underline{v}})})_{|t|}$ is a valued-profile corr. to

$$(\zeta_{2l}^{i_1} \cdot \underline{\underline{q}}_{\underline{v}}^{1^2}, \ \zeta_{2l}^{i_2} \cdot \underline{\underline{q}}_{\underline{v}}^{2^2}, \ \dots, \ \zeta_{2l}^{i_{l^*}} \cdot \underline{\underline{q}}_{\underline{v}}^{(l^*)^2})$$

— where $\zeta_{2l}^{i_j}$ is a generator of $\boldsymbol{\mu}_{2l}$.

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$$\mathcal{I} \stackrel{\text{def}}{=} (2p)^{-1} \cdot \log_{\overline{k}}(\mathcal{O}^{\times}_{k}) \subseteq k = \{0\} \cup (\mathcal{O}_k)^{\text{gp}}$$

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Note: $\mathcal{O}^{\times\boldsymbol{\mu}}_{k} \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{I} \otimes \mathbb{Q}$; $\mathcal{O}^{\times\boldsymbol{\mu}}_{k} \otimes (2p)^{-1} \xrightarrow{\sim} \mathcal{I}$

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$$\begin{array}{ccc} \mathcal{O}^{\rhd}_{\overline{k}}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}) & \xrightarrow{\sim} & \mathcal{O}^{\rhd}_{\overline{k}}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}}) \\ \mathrm{Kum} \uparrow & & \uparrow \mathrm{Kum} \\ \mathcal{O}^{\rhd}_{\overline{k}} & \xrightarrow{\mathfrak{log}} & \mathcal{O}^{\rhd}_{\overline{k}} \end{array}$$

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$\Rightarrow$ We use inclusions $\mathcal{O}^{\rhd}_{k} \subseteq \mathcal{I} \supseteq \log_{\overline{k}}(\mathcal{O}^{\times}_{k})$. In particular, "$\mathcal{I}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}})$" contains the images of Kum assoc. to both the dom/codom of $\mathfrak{log}$ ("upper semi-commutativity"). $\Rightarrow$ (Ind3)

Theorem (An Approximate Statement of the Main Theorem of IUT)

For a general initial Θ*-data* $(\overline{F}/F,\ E,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\mathbb{V}}^{\mathrm{bad}},\ \underline{\epsilon})$*,*
$^{\exists}$*suitable multiradial algorithm whose output data consist of the following three objects* $\curvearrowleft$ *(Ind1), (Ind2), (Ind3)*

- *the collection of log-shells* $\{\mathcal{I}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$*;*
- *the theta values* $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\} \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}_{\underline{v}}$*;*
- *the number field* $F_{\mathrm{mod}} \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}}(\mathcal{I}_{\underline{v}} \otimes \mathbb{Q})$*.*

Moreover, this alg'm is compatible w/ the Θ*-link* ${}^{\dagger}\mathcal{HT} \longrightarrow {}^{\ddagger}\mathcal{HT}$*.*

Log-links

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$: a place lying over $p \in \mathbb{Z}$

${}^{\dagger}\mathfrak{F} = \{{}^{\dagger}\mathcal{F}_{\underline{w}}\}_{\underline{w} \in \underline{\mathbb{V}}}$: an $\mathcal{F}$-prime-strip. In particular,

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Note: $\mathcal{O}^{\times}_{\overline{k}} \otimes \mathbb{Q} \xrightarrow{\sim} \overline{k}$; $\boxtimes \rightsquigarrow \boxplus$

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Observe: The underlying $\mathcal{D}$-prime-strip of $\mathfrak{log}({}^{\dagger}\mathfrak{F})$ coincides with the underlying $\mathcal{D}$-prime-strip of ${}^{\dagger}\mathfrak{F}$.

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${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, ${}^{\dagger\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$: $\Theta^{\pm\mathrm{ell}}\mathrm{NF}$-Hodge theaters

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<u>Note</u>: ${}^{\forall}$isom. $\Xi : {}^{\dagger\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ induces an isom.
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We shall write

$$ {}^{\dagger\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\mathfrak{log}}{\to} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} $$

and refer to as the (full) $\mathfrak{log}$-link the collection

$$ \{\mathfrak{log}({}^{\dagger\dagger}\mathfrak{F}_{\square}) \underset{\Xi}{\xrightarrow{\sim}} {}^{\dagger}\mathfrak{F}_{\square}\}_{\square \in T \cup J \cup \{\succ\} \cup \{>\},\ \Xi} $$

— where we consider all isoms $\Xi : {}^{\dagger\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

Log-shells

We shall refer to

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \overset{\text{def}}{=} \frac{1}{2p} \cdot \mathrm{Im}\left(\left({}^\dagger\overline{\mathcal{O}}_{\underline{v}}^{\times}\right)^{{}^\dagger\Pi_{\underline{v}}} \hookrightarrow {}^\dagger\overline{\mathcal{O}}_{\underline{v}}^{\times} \twoheadrightarrow \widetilde{k}({}^\dagger\mathcal{F}_{\underline{v}})\right) \subseteq \widetilde{k}({}^\dagger\mathcal{F}_{\underline{v}})^{{}^\dagger\Pi_{\underline{v}}}$$

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${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$: a $\mathcal{D}$-prime-strip (In particular, ${}^\dagger\mathcal{D}_{\underline{v}} \approx {}^\dagger\Pi_{\underline{v}}$.)

Log-shells

We shall refer to

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \overset{\text{def}}{=} \frac{1}{2p} \cdot \mathrm{Im}(({}^\dagger\overline{\mathcal{O}}_{\underline{v}}^{\times})^{{}^\dagger\Pi_{\underline{v}}} \hookrightarrow {}^\dagger\overline{\mathcal{O}}_{\underline{v}}^{\times} \twoheadrightarrow \widetilde{k}({}^\dagger\mathcal{F}_{\underline{v}})) \subseteq \widetilde{k}({}^\dagger\mathcal{F}_{\underline{v}})^{{}^\dagger\Pi_{\underline{v}}}$$

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$$ {}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}} \;\approx\; ({}^{\dagger}G_{\underline{v}} \curvearrowright {}^{\dagger}\overline{\mathcal{O}}^{\times\boldsymbol{\mu}}_{\underline{v}},\ \{\mathcal{I}^{\kappa}_{H}\}_{H\subseteq {}^{\dagger}G_{\underline{v}}}) $$

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Then since we can reconstruct "p" (from ${}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}}$), and, moreover,

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\widetilde{k}_{+}({}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\boldsymbol{\mu}}) \;\stackrel{\text{def}}{=}\; {}^{\dagger}\overline{\mathcal{O}}_{\underline{v}}^{\times\boldsymbol{\mu}} \otimes \mathbb{Q} \;\stackrel{\sim}{\to}\; {}^{\dagger}\overline{\mathcal{O}}_{\underline{v}}^{\times} \otimes \mathbb{Q},
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$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F} & \rightsquigarrow & {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}} \\ \wr\downarrow & & \wr\downarrow \\ {}^{\dagger}\mathfrak{D} & \rightsquigarrow & {}^{\dagger}\mathfrak{D}^{\vdash} \end{array}$$

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— where $(\mathrm{Ind2})_{\underline{v}} = \mathrm{Ism}_{\underline{v}} \ (= \mathrm{Aut}^{\times\boldsymbol{\mu}\text{-Kum}}_{G_{\underline{v}}}(\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}))$.

Let $\mathcal{I} \in \{\mathcal{I}_{\dagger\mathcal{F}_{\underline{v}}},\ \mathcal{I}_{\dagger\mathcal{D}_{\underline{v}}}\}$. Then, using the field str. on $\mathcal{I} \otimes \mathbb{Q}$ ($\cong K_{\underline{v}}$), for any cpt op. $A \subseteq \mathcal{I} \otimes \mathbb{Q}$, we can define the volume $\mu_{\underline{v}}(A) \in \mathbb{R}_{>0}$ satisfying the following:

- $A \cap B = \emptyset \implies \mu_{\underline{v}}(A \cup B) = \mu_{\underline{v}}(A) + \mu_{\underline{v}}(B)$.
- $x \in \mathcal{I} \otimes \mathbb{Q}$ ($\cong K_{\underline{v}}$) $\implies \mu_{\underline{v}}(x + A) = \mu_{\underline{v}}(A)$.
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By applying the theory of [AbsTopIII], we can also reconstruct $\mu_{\underline{w}}^{\log}$ on $\mathcal{I}_{\dagger \mathcal{F}_{\underline{w}}^{\vdash} \times \boldsymbol{\mu}} \otimes \mathbb{Q}$ and $\mathcal{I}_{\dagger \mathcal{D}_{\underline{w}}^{\vdash}} \otimes \mathbb{Q}$ (s.t. they are "compatible").

Processions

Note: The (re)const'n of labels $j \in \mathbb{F}_l^{\circledast} = \mathbb{F}_l^{\times}/\{\pm 1\} = \{1, \dots, l^{\circledast}\}$ dep. on "$\Pi_{\underline{v}}$" which is not "shared" by an alien arith. hol str. So

Processions

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$\Longrightarrow$ We consider a procession, i.e., the diag. of inclusions of fin. sets

$$\mathbb{S}_1^{\pm} \hookrightarrow \mathbb{S}_2^{\pm} \hookrightarrow \cdots \hookrightarrow \mathbb{S}_{j+1}^{\pm} \hookrightarrow \cdots \hookrightarrow \mathbb{S}_{l^{\pm}}^{\pm}$$

— where we write $\mathbb{S}_{j+1}^{\pm} = \{0, 1, \ldots, j\}$, $l^{\pm} \overset{\text{def}}{=} l^{\circledast} + 1$, and we think of each of these sets as being subj. to arbitrary permutation automs.

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$\Longrightarrow$ If one allows $j = 0, \ldots, l^{\circledast}$ to vary, then this trick reduces the resulting label indet. from a total of possibilities $(l^{\pm})^{l^{\pm}}$ to $l^{\pm}!$

Local tensor packets

$\{{}^{\dagger}\mathfrak{F}_{|t|}\}_{|t| \in |T|}$: a "capsule" of $\mathcal{F}$-prime-strips

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$$\Longrightarrow \ (\{^{\dagger}\mathfrak{F}_{|t|}\}_{|t| \in \mathbb{S}_1^{\pm}} \hookrightarrow \cdots \hookrightarrow \{^{\dagger}\mathfrak{F}_{|t|}\}_{|t| \in \mathbb{S}_j^{\pm}} \hookrightarrow \cdots \hookrightarrow \{^{\dagger}\mathfrak{F}_{|t|}\}_{|t| \in \mathbb{S}_{l^{\pm}}^{\pm}})$$

Let us define the local holomorphic tensor packets as follows:

Local tensor packets

$\{{}^\dagger\mathfrak{F}_{|t|}\}_{|t|\in|T|}$: a "capsule" of $\mathcal{F}$-prime-strips

$$\Longrightarrow\ (\{{}^\dagger\mathfrak{F}_{|t|}\}_{|t|\in\mathbb{S}^\pm_1} \hookrightarrow \cdots \hookrightarrow \{{}^\dagger\mathfrak{F}_{|t|}\}_{|t|\in\mathbb{S}^\pm_j} \hookrightarrow \cdots \hookrightarrow \{{}^\dagger\mathfrak{F}_{|t|}\}_{|t|\in\mathbb{S}^\pm_{l^\pm}})$$

Let us define the local holomorphic tensor packets as follows:

For $|t| \in |T|$, $\underline{v}|v_\mathbb{Q} \in \mathbb{V}(\mathbb{Q})$, $1 \le j \le l^\pm \stackrel{\mathrm{def}}{=} l^* + 1$,

$$\widetilde{k}^\otimes({}^\dagger\mathcal{F}_{|t|,v_\mathbb{Q}}) \stackrel{\mathrm{def}}{=} \prod_{\underline{\mathbb{V}}\ni\underline{w}|v_\mathbb{Q}} \widetilde{k}({}^\dagger\mathcal{F}_{|t|,\underline{w}});$$

$$\widetilde{k}^\otimes({}^\dagger\mathcal{F}_{\mathbb{S}^\pm_j,v_\mathbb{Q}}) \stackrel{\mathrm{def}}{=} \bigotimes_{|t|\in\mathbb{S}^\pm_j} \widetilde{k}^\otimes({}^\dagger\mathcal{F}_{|t|,v_\mathbb{Q}});$$

$$\widetilde{k}^\otimes({}^\dagger\mathcal{F}_{\mathbb{S}^\pm_j,\underline{v}}) \stackrel{\mathrm{def}}{=} \widetilde{k}^\otimes({}^\dagger\mathcal{F}_{\mathbb{S}^\pm_{j-1},v_\mathbb{Q}}) \otimes \widetilde{k}^\otimes({}^\dagger\mathcal{F}_{|j-1|,\underline{v}}) \subseteq \widetilde{k}^\otimes({}^\dagger\mathcal{F}_{\mathbb{S}^\pm_j,v_\mathbb{Q}}).$$

By replacing "$\widetilde{k}$" by "$\mathcal{I}$", we obtain the compact submodules

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{F}_{|t|,v_{\mathbb{Q}}}) \ \subseteq\ \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{F}_{|t|,v_{\mathbb{Q}}}); \quad \mathcal{I}^{\otimes}({}^{\dagger}\mathcal{F}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \ \subseteq\ \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{F}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}});$$

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{F}_{\mathbb{S}^{\pm}_{j},\underline{v}}) \ \subseteq\ \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{F}_{\mathbb{S}^{\pm}_{j},\underline{v}}).$$

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By considering the $\mathbb{Q}$-spans of them, we obtain the $\mathbb{Q}_p$-vector spaces

$$(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{F}_{|t|,v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{F}_{|t|,v_{\mathbb{Q}}}); \quad (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{F}_{\mathbb{S}_j^{\pm},v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{F}_{\mathbb{S}_j^{\pm},v_{\mathbb{Q}}});$$

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Note: We can construct étale-like versions

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{D}_{|t|,v_{\mathbb{Q}}}) \ \subseteq\ (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{D}_{|t|,v_{\mathbb{Q}}}) \ \subseteq\ \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{D}_{|t|,v_{\mathbb{Q}}});$$

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{D}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \ \subseteq\ (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{D}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \ \subseteq\ \widetilde{k}^{\otimes}({}^{\dagger}\mathcal{D}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}});$$

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$$\Longrightarrow\ (\{{}^{\dagger}\mathfrak{F}_{|t|}^{\vdash\times\boldsymbol{\mu}}\}_{|t|\in\mathbb{S}_1^{\pm}} \hookrightarrow \cdots \hookrightarrow \{{}^{\dagger}\mathfrak{F}_{|t|}^{\vdash\times\boldsymbol{\mu}}\}_{|t|\in\mathbb{S}_j^{\pm}} \hookrightarrow \cdots \hookrightarrow \{{}^{\dagger}\mathfrak{F}_{|t|}^{\vdash\times\boldsymbol{\mu}}\}_{|t|\in\mathbb{S}_{l^{\pm}}^{\pm}})$$

Let us define the local mono-analytic tensor packets as follows:

$\{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{|t|}\}_{|t|\in|T|}$: a "capsule" of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$$\Longrightarrow\ (\{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{|t|}\}_{|t|\in\mathbb{S}^{\pm}_{1}} \hookrightarrow \cdots \hookrightarrow \{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{|t|}\}_{|t|\in\mathbb{S}^{\pm}_{j}} \hookrightarrow \cdots \hookrightarrow \{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{|t|}\}_{|t|\in\mathbb{S}^{\pm}_{l^{\pm}}})$$

Let us define the local mono-analytic tensor packets as follows:

For $|t|\in|T|$, $\underline{v}|v_{\mathbb{Q}}\in\mathbb{V}(\mathbb{Q})$, $1\le j\le l^{\pm}$,

$$\widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}}) \stackrel{\text{def}}{=} \prod_{\underline{\mathbb{V}}\ni\underline{w}|v_{\mathbb{Q}}} \widetilde{k}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,\underline{w}});$$

$$\widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \stackrel{\text{def}}{=} \bigotimes_{|t|\in\mathbb{S}^{\pm}_{j}} \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}});$$

$$\widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},\underline{v}}) \stackrel{\text{def}}{=} \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j-1},v_{\mathbb{Q}}}) \otimes \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|j-1|,\underline{v}}) \subseteq \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}).$$

By replacing "$\widetilde{k}_{+}$" by "$\mathcal{I}$", we obtain the compact submodules

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}}) \ \subseteq \ \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}}); \quad \mathcal{I}^{\otimes}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \ \subseteq \ \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}});$$

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By considering the $\mathbb{Q}$-spans of them, we obtain the $\mathbb{Q}_p$-vector spaces

$$(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{|t|,v_{\mathbb{Q}}}); \quad (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}});$$

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Note: We can construct étale-like versions

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{D}^{\vdash}_{|t|,v_{\mathbb{Q}}}) \subseteq (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{D}^{\vdash}_{|t|,v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{D}^{\vdash}_{|t|,v_{\mathbb{Q}}});$$

$$\mathcal{I}^{\otimes}({}^{\dagger}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \subseteq (\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}}) \subseteq \widetilde{k}^{\otimes}_{+}({}^{\dagger}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j},v_{\mathbb{Q}}});$$

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Local logarithmic Gaussian procession monoids

We consider the infinite chain of $\mathfrak{log}$-links

$$\ldots \overset{\mathfrak{log}}{\to} {}^{-1\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{-0\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{1\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \ldots$$

which determines

$$\ldots \overset{\sim}{\to} {}^{-1\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{-0\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{1\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} \ldots .$$

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Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, $n \in \mathbb{Z}$.

<u>Recall</u>: We have the Frobenius-like Gaussian monoids

$$\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subseteq {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subseteq \prod_{|t|\in T^{*}} \overline{\mathcal{O}}^{\rhd}_{\underline{v}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}})_{|t|}.$$

We shall write

$$ {}^{\dagger n}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}, \quad {}^{\dagger n}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} $$

and refer to as the Frobenius-like local Logarithmic Gaussian Procession monoids the images of $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}})$, ${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ via

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$$ \prod_{|t|\in T^{*}} \overline{\mathcal{O}}^{\rhd}_{\underline{v}}({}^{\dagger n}\underline{\underline{\mathcal{F}}}_{\underline{v}})_{|t|} \overset{\sim}{\to} \prod_{|t|\in T^{*}} \overline{\mathcal{O}}^{\rhd}_{\underline{v}}({}^{\dagger n}\mathcal{F}_{|t|,\underline{v}})_{|t|} \overset{\sim}{\leftarrow} \prod_{|t|\in T^{*}} \mathcal{O}^{\rhd}_{\widetilde{k}({}^{\dagger n-1}\mathcal{F}_{|t|,\underline{v}})} $$

$$ \subseteq \prod_{|t|\in T^{*}} \widetilde{k}({}^{\dagger n-1}\mathcal{F}_{|t|,\underline{v}}) \hookrightarrow \prod_{j\in J} \widetilde{k}^{\otimes}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) $$

— where "$\overset{\sim}{\to}$" arises from the definition of Hodge theaters;

"$\overset{\sim}{\leftarrow}$" arises from the $\mathfrak{log}$-link ${}^{n-1\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{n\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

"$\hookrightarrow$" arises from $T^{*} \overset{\sim}{\to} J$ and "$(-) \mapsto 1 \otimes (-)$".

We shall write

$$ {}^{\circ\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} $$

for the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}\mathrm{NF}$ theater (det'd up to isom.) obtained by identifying the infinite chain of (full-poly) isomorphisms

$$ \cdots \overset{\sim}{\to} {}^{-1\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{-0\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{1\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} \cdots . $$

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Then, by applying a similar const. to the étale-like Gaussian monoids

$$ \Psi_{\mathrm{gau}}({}^{\dagger\circ}\mathcal{D}_{\succ,\underline{v}}) \subseteq {}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger\circ}\mathcal{D}_{\succ,\underline{v}}) \subseteq \prod_{|t|\in T^{*}} \overline{\mathcal{O}}^{\triangleright}_{\underline{v}}({}^{\dagger\circ}\mathcal{D}_{\succ,\underline{v}})_{|t|}, $$

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we obtain étale-like local Logarithmic Gaussian Procession monoids

$$ {}^{\dagger\circ}\Psi_{\mathcal{D}_{\mathrm{LGP}},\underline{v}}, \quad {}^{\dagger\circ}_{\infty}\Psi_{\mathcal{D}_{\mathrm{LGP}},\underline{v}}. $$

$\mathfrak{log}$-Kummer correspondences (LGP monoids)

For each $n \in \mathbb{Z}$, we have the Kummer isomorphisms

$$^{\dagger n}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \xrightarrow{\sim} {}^{\dagger\circ}\Psi_{\mathcal{D}_{\mathrm{LGP}},\underline{v}}, \quad {}^{\dagger n}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \xrightarrow{\sim} {}^{\dagger\circ}_{\infty}\Psi_{\mathcal{D}_{\mathrm{LGP}},\underline{v}}.$$

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$^{\dagger n}\Psi^{\times G}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \subseteq {}^{\dagger n}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$: the Gal-inv. of the group of units $(\cong \mathcal{O}^{\times}_{K_{\underline{v}}})$

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Note: ${}^{\dagger n}\Psi^{\times G}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$ and ${}^{\dagger n}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$ act on the tensor packets

$$ \prod_{j\in J}(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) \xrightarrow{\sim} \prod_{j\in J}(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}}). $$

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<u>Note</u>: $^{\dagger n}\Psi^{\times G}_{\mathcal{F}_{\mathrm{LGP},\underline{v}}}$ and $^{\dagger n}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP},\underline{v}}}$ act on the tensor packets

$$\prod_{j\in J}(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) \xrightarrow{\sim} \prod_{j\in J}(\mathcal{I}^{\otimes})^{\mathbb{Q}}({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}}).$$

We want to make these actions "invariant w.r.t. the action $+1 \curvearrowright \mathbb{Z}$".

Write $(\mathcal{I}^{\otimes})^{\mathbb{Q}}_{J}(-) \overset{\text{def}}{=} \prod_{j \in J}(\mathcal{I}^{\otimes})^{\mathbb{Q}}(-)$. Let $\Box \in \{\times G, \perp\}$.

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$$
\begin{array}{ccccccc}
{}^{\dagger n}\Psi^{\Box}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} & & & & & & \\
\curvearrowright & & & & & & \\
(\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n+1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathrm{Kum} \downarrow \wr & & \mathrm{Kum} \downarrow \wr & & \mathrm{Kum} \downarrow \wr & & \\
(\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}}) & = & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}}) & = & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}}) & = & \cdots
\end{array}
$$

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$$
\begin{array}{ccccccc}
{}^{\dagger n}\Psi^{\square}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} & & & & & & \\
\curvearrowright & & & & & & \\
(\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger n+1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},\underline{v}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathrm{Kum}\downarrow\wr & & \mathrm{Kum}\downarrow\wr & & \mathrm{Kum}\downarrow\wr & & \\
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\end{array}
$$

In IUT, we consider the $\mathfrak{log}$-Kummer correspondence

$$
\left\{\mathrm{Kum}\circ\mathfrak{log}^m({}^{\dagger n}\Psi^{\square}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}) \curvearrowright (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}})\right\}_{n\in\mathbb{Z},\ m\in\mathbb{Z}_{\geq 0}}
$$

which is "invariant" w.r.t. the action $\mathbb{Z}\ni n\mapsto n+1\in\mathbb{Z}$.

Note:

- Suppose that $\square = \perp$. Then the only portions of these actions that are possibly related to one another via these $\mathfrak{log}$-links are the indeterminacies w. r. t. multiplication by roots of unity in the domains of the $\mathfrak{log}$-links (cf. const. mult. rigidity).

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- Suppose that $\Box = \times G$. Then we only know

$$\bigcup_{n\in\mathbb{Z},\ m\in\mathbb{Z}_{\geq 0}} \mathrm{Kum} \circ \mathfrak{log}^m({}^{\dagger n}\Psi^{\times G}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}) \ \subseteq\ (\mathcal{I}^{\otimes})^{\mathbb{Q}}_J({}^{\dagger\circ}\mathcal{D}_{\succ,\mathbb{S}^{\pm}_{j+1},\underline{v}})$$

(cf. "upper semi-commutativity")

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(cf. "upper semi-commutativity") $\Longrightarrow$ (Ind3)!

Case of number fields

Note: By our assumption, $C = [X/\{\pm 1\}]$ descends to the hyperbolic orbicurve $C_{F_{\mathrm{mod}}}$ over F_{mod}.

$$\mathcal{D}^{\circledcirc} \approx \pi_1^{\text{ét}}(\underline{C}_K), \ \mathcal{D}^{\circledast} \approx \pi_1^{\text{ét}}(C_{F_{\mathrm{mod}}}) \implies \mathcal{D}^{\circledcirc} \to \mathcal{D}^{\circledast}$$

$$S_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} [\mathrm{Spec}(\mathcal{O}_K)/\mathrm{Gal}(K/F_{\mathrm{mod}})]$$

$\mathcal{F}^{\circledcirc}$ (resp. $\mathcal{F}^{\circledast}$): the Frobenioid whose objects are pairs $(X, \mathcal{L})$
where X is a fét cov. of $\underline{C}_K$ (resp. $C_{F_{\mathrm{mod}}}$);
$\mathcal{L}$ is an arith. line bdl over $\mathrm{Nor}(X/S_{\mathrm{mod}})$

$\mathcal{F}^{\circledast}_{\mathrm{mod}}$: the Frobenioid whose objects are pairs $(S_{\mathrm{mod}}, \mathcal{L})$
$\mathcal{L}$ is an arith. line bdl over S_{mod}

Theorem

$^{\exists}$*functional algorithm*

$$^{\dagger}\mathcal{D}^{\circledcirc} \quad \longmapsto \quad F_{\mathrm{mod}}(^{\dagger}\mathcal{D}^{\circledcirc})$$

corresponding to the field F_{mod}. *Moreover,* $^{\exists}$*functorial algorithm for constructing the Kummer isomorphism*

$$(^{\dagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\dagger}\mathcal{F}^{\circledast}) \quad \longmapsto \quad F_{\mathrm{mod}}(^{\dagger}\mathcal{F}^{\circledast})^{\times} \stackrel{\sim}{\to} F_{\mathrm{mod}}(^{\dagger}\mathcal{D}^{\circledcirc})^{\times}.$$

Theorem

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Idea:

- Evaluation of a "κ-coric function" at various pts $\rightsquigarrow$ $F^\times_{\mathrm{mod}}$
- Reconstruct the decom. gps $\subseteq \pi_1^{\text{ét}}(C_{F_{\mathrm{mod}}})$ assoc. to various pts by applying the theory of Belyi cuspidalization
- An elementary equality $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$

Global tensor packets

Note: ${}^{\dagger}F_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} F_{\mathrm{mod}}({}^{\dagger}\mathcal{F}^{\circledast}) \cup \{0\}$ admits a nat'l str. of field.

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$\Longrightarrow$ We obtain the global tensor packet

$$(^{\dagger}F_{\mathrm{mod}})^{\otimes}_{\mathbb{S}_j^{\pm}} \overset{\mathrm{def}}{=} \bigotimes_{|t| \in \mathbb{S}_j^{\pm}} (^{\dagger}F_{\mathrm{mod}})_{|t|}$$

— where $(^{\dagger}F_{\mathrm{mod}})_{|t|} = (^{\dagger}F_{\mathrm{mod}})_j$ (if $|t| \in T^{⋇} \cong J$); $(^{\dagger}F_{\mathrm{mod}})_{\langle J \rangle}$ (if $|t| = 0$).

Two natural ways to approach the construction of $\mathcal{F}^{\circledast}_{\mathbf{mod}}$

- $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ (rational function torsor version)

An object $\mathcal{T} = (T, \{t_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$ of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ consists of a collection

(a) an $F^{\times}_{\mathrm{mod}}$-torsor T;

(b) for each $\underline{v} \in \underline{\mathbb{V}}$, the trivialization $t_{\underline{v}}$ of the torsor "$T_{\underline{v}}$" obt'd from T

subj. to a certain condition "$^{\exists}t \in T$ s.t. $t_{\underline{v}}$ coincides w/ ..."

- $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ (local fractional ideal version)

An object $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ consists of a collection of "fractional ideals" $J_{\underline{v}} \subseteq K_{\underline{v}}$ s.t. $J_{\underline{v}} = \mathcal{O}_{K_{\underline{v}}}$ for a.a. $\underline{v} \in \underline{\mathbb{V}}$

We have nat'l isoms of Frobenioids

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}} \overset{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{mod}} \overset{\sim}{\to} \mathcal{F}^{\circledast}_{\mathrm{MOD}}.$$

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- The construction of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ depends only on the multiplicative structure of $F^{\times}_{\mathrm{mod}}$.

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<u>Note</u>:

- The construction of $\mathcal{F}_{\mathrm{MOD}}^{\circledast}$ depends only on the multiplicative structure of $F_{\mathrm{mod}}^{\times}$.
  $\Longrightarrow$ "not interfere" in the $\mathfrak{log}$-Kummer corr. (cf. below)

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  $\Longrightarrow$ "interfere" in the $\mathfrak{log}$-Kummer corr. (cf. $(\mathrm{Ind}3)$)
  $\Longrightarrow$ but, suited to the explicit comp. by means of log-volumes

$\mathfrak{log}$-Kummer correspondences (number fields)

We consider the infinite chain of $\mathfrak{log}$-links

$$\dots \overset{\mathfrak{log}}{\to} {}^{-1\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{-0\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} {}^{1\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\mathfrak{log}}{\to} \dots .$$

Let $n \in \mathbb{Z}$, $1 \leq j \leq l^{*}$.

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$$({}^{\dagger n}F_{\mathrm{mod}})_j \hookrightarrow ({}^{\dagger n}F_{\mathrm{mod}})^{\otimes}_{\mathbb{S}^{\pm}_{j+1}} \hookrightarrow \prod_{v_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})} \widetilde{k}^{\otimes}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}).$$

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for this image. $\rightsquigarrow$ Using $({}^{\dagger n}F_{\mathrm{MOD}})_j$, we obtain $({}^{\dagger n}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$

Using $({}^{\dagger n}F_{\mathfrak{mod}})_{|j|}$, together w/ the integral structure

$$\mathcal{O}_{\widetilde{k}^{\otimes}({}^{\dagger n-1}\mathcal{F}_{|j|,\underline{v}})} \hookrightarrow \widetilde{k}^{\otimes}({}^{\dagger n-1}\mathcal{F}_{|j|,\underline{v}}) \hookrightarrow \widetilde{k}^{\otimes}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1},v_{\mathbb{Q}}}),$$

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Note: We have étale-like versions and Kummer isom.

$$({}^{\dagger n}F_{\mathrm{MOD}})_j \xrightarrow{\sim} ({}^{\dagger\circ}F_{\mathcal{D}_{\mathrm{MOD}}})_j, \quad ({}^{\dagger n}F_{\mathfrak{mod}})_j \xrightarrow{\sim} ({}^{\dagger\circ}F_{\mathcal{D}_{\mathfrak{mod}}})_j$$

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Note: Thanks to the integral str., we can compute the degrees of arith. line bdls "$\in$" $({}^{\dagger n}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$, $({}^{\dagger\circ}\mathcal{F}^{\circledast}_{\mathcal{D}_{\mathfrak{mod}}})_j$ by means of the log-volumes.

Write $(\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}(-) \stackrel{\text{def}}{=} \prod_{v_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})} (\mathcal{I}^{\otimes})^{\mathbb{Q}}(-)$.

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$$({}^{\dagger n}F_{\mathrm{MOD}})^{\times}_{j}$$

$$\curvearrowright$$

$$\begin{array}{ccccccc}
(\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger n-1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger n}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger n+1}\mathcal{F}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & \xrightarrow{\mathfrak{log}} & \cdots \\
\mathrm{Kum} \downarrow \wr & & \mathrm{Kum} \downarrow \wr & & \mathrm{Kum} \downarrow \wr & & \\
(\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & = & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & = & (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) & = & \cdots
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$$
\begin{array}{ccccccc}
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\curvearrowright & & & & & & \\
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\end{array}
$$

In IUT, we consider the $\mathfrak{log}$-Kummer correspondence

$$
\left\{ \mathrm{Kum} \circ \mathfrak{log}^{m}(({}^{\dagger n}F_{\mathrm{MOD}})^{\times}_{j}) \ \curvearrowright \ (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) \right\}_{n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geq 0}}
$$

which is "invariant" w.r.t. the action $\mathbb{Z} \ni n \mapsto n+1 \in \mathbb{Z}$.

Note:

- The only portions of these actions that are possibly related to one another via these $\mathfrak{log}$-links are the indeterminacies w. r. t. multiplication by roots of unity in the domains of the $\mathfrak{log}$-links — cf. the following fact:

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$\Longrightarrow$ Indeterminacies at n that correspond — via $\mathfrak{log}$ — to "addition by zero" at $n+1$ $\Longrightarrow$ non-interference!

$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links

Let us recall the notion of a $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link (cf. [IUT2]):

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\to} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}; \quad {}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$$

(a) (loc. unit gps) $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \mapsto G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$

(b) (loc. val. gps) $(\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j=1,2,\dots,l^*})^{\mathbb{N}} \mapsto \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$ (if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$)

(c) (glob. val. gps) glob. real'd Frob. $\mapsto$ glob. real'd Frob.

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We consider two infinite chains of $\mathfrak{log}$-links

$$\dots \xrightarrow{\mathfrak{log}} {}^{\dagger -1}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger 0}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger 1}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} \dots$$

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Replacing the data in the left-hand side of (a), (b) (resp. (c)) by the data arise from $\{{}^{\dagger 0}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ (resp. $({}^{\dagger 0}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} ({}^{\dagger 0}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$), we obtain the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link

$$
{}^{\dagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}}{\to} {}^{\ddagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}; \quad {}^{\dagger 0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{\ddagger 0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}.
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{}^{\dagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}}{\to} {}^{\ddagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}; \quad {}^{\dagger 0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \xrightarrow{\sim} {}^{\ddagger 0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}.
$$

Note: Then we have

- objects of (c) in ${}^{\dagger 0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$ det'd by "$\{\underline{\underline{q}}_{\underline{w}}^{j^2}\}_{j=1,2,\dots,l^*;\ \underline{w}\in\underline{\mathbb{V}}^{\mathrm{bad}}}$" which we shall refer to as Θ-pilot objects
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$\Longrightarrow$ The $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link maps Θ-pilots object to q-pilot objects.

Multiradial algorithms via LGP-monoids/Frobenioids

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(a) For $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})$, $1 \leq j \leq l^{\pm}$, the mono-an. ét-like log-shells

$$\mathcal{I}^{\otimes}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j}, v_{\mathbb{Q}}}), \quad \mathcal{I}^{\otimes}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j}, \underline{v}}),$$

and the (procession-normalized) log-volumes on them.

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(b) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the ét-like splitting monoid

$${}^{\dagger\circ}\Psi^{\perp}_{\mathcal{D}_{\mathrm{LGP}}, \underline{v}} \curvearrowright \left((\mathcal{I}^{\otimes})^{\mathbb{Q}}_{J}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, \underline{v}}) \xrightarrow{\sim} \right) (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{J}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j+1}, \underline{v}})$$

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(c) For $1 \leq j \leq l^{\pm}$, the ét-like number field

$$({}^{\dagger\circ}F_{\mathcal{D}_{\mathrm{MOD}}})_j = ({}^{\dagger\circ}F_{\mathcal{D}_{\mathfrak{mod}}})_j \curvearrowright \left((\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}_{\succ, \mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}) \stackrel{\sim}{\to} \right) (\mathcal{I}^{\otimes})^{\mathbb{Q}}_{\mathbb{V}}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_{j+1}, v_{\mathbb{Q}}}),$$

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- (Ind1) — which arises from the aut of proc. of $\mathcal{D}^{\vdash}$-prime-strips
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(ii) ($\mathfrak{log}$-Kummer correspondence) For $n \in \mathbb{Z}$, we have the Kum. isoms

(a) For $\underline{\mathbb{V}} \ni \underline{v} | v_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})$, $1 \leq j \leq l^{\pm}$,

$$\mathcal{I}^{\otimes}({}^{\dagger n}\mathcal{F}_{\mathbb{S}^{\pm}_j, v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}^{\otimes}({}^{\dagger n}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_j, v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}^{\otimes}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_j, v_{\mathbb{Q}}}),$$

$$\mathcal{I}^{\otimes}({}^{\dagger n}\mathcal{F}_{\mathbb{S}^{\pm}_j, \underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\otimes}({}^{\dagger n}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\mathbb{S}^{\pm}_j, \underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\otimes}({}^{\dagger\circ}\mathcal{D}^{\vdash}_{\mathbb{S}^{\pm}_j, \underline{v}})$$

which are comp. w/ the respective log-volumes

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(b) For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

$$^{\dagger n}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP},\underline{v}}} \xrightarrow{\sim} {}^{\dagger\circ}\Psi^{\perp}_{\mathcal{D}_{\mathrm{LGP},\underline{v}}}$$

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(c) For $1 \leq j \leq l^{\pm}$,

$$({}^{\dagger n}F_{\mathrm{MOD}})_j \xrightarrow{\sim} ({}^{\dagger\circ}F_{\mathcal{D}_{\mathrm{MOD}}})_j, \quad ({}^{\dagger n}F_{\mathfrak{mod}})_j \xrightarrow{\sim} ({}^{\dagger\circ}F_{\mathcal{D}_{\mathfrak{mod}}})_j$$

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Note:

- As one varies $n \in \mathbb{Z}$, the various isoms of (b) and of the first line of (c) are compatible. ($\Longrightarrow$ compatibility concerning "MOD")
- By allowing (Ind3), as one varies $n \in \mathbb{Z}$, the various isoms of (a) are compatible.

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(iii) The isoms of (ii) are "comp." w/ ${}^{\dagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}}{\rightarrow} {}^{\ddagger 0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

Log-volume estimates for Θ-pilot objects

Corollary

Write

$$-|\log(\underline{\underline{\Theta}})| \in \mathbb{R} \cup \{\infty\}$$

for the (process.-normalized, mono-an.) log-volume of the "holomorphic hull" of the union of the possible images of a Θ-pilot object, rel. to the relevant Kum. isoms, in the multira'l rep'n of the previous (i), which we regard as sub. to (Ind1), (Ind2), (Ind3);

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Log-volume estimates for Θ-pilot objects

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Then it holds that $-|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$, *and* $-|\log(\underline{\underline{\Theta}})| \geq -|\log(\underline{\underline{q}})|$.